

# QUILLEN THEOREMS $B_n$ FOR HOMOTOPY PULLBACKS OF $(\infty, k)$ -CATEGORIES

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**ABSTRACT.** We extend the Quillen Theorem  $B_n$  for *homotopy fibers* of Dwyer et al. to similar results for *homotopy pullbacks* and note that these results imply similar results for zigzags in the categories of *relative categories* and *k-relative categories*, not only with respect to their *Reedy* structures but also their *Rezk* structure, which turns them into models for the theories of  $(\infty, 1)$ - and  $(\infty, k)$ -categories respectively.

Our main tool for proving this are the *sharp maps* of Hopkins and Rezk which, because of their fibration-like properties, we prefer to call *fibrillations*.

## 0. INTRODUCTION

### 0.1. Background.

- (i) In [Q, §1] Quillen proved his Theorem B which states that, for a functor  $f: \mathbf{X} \rightarrow \mathbf{Z}$  and an object  $Z \in \mathbf{Z}$ , the rather simple *over category*  $(f \downarrow Z)$  is a *homotopy fiber* of  $f$  if  $f$  has a certain *property*  $B_1$ .
- (ii) This was generalized in [DKS, §6] where it was shown that increasingly weaker *properties*  $B_n$  ( $n > 1$ ) allowed for increasingly less simple description of these homotopy fibers as *n-arrow overcategories*  $(f \downarrow_n Z)$ .
- (iii) It was also noted that a sufficient condition for a functor  $f: \mathbf{X} \rightarrow \mathbf{Z}$  to have property  $B_n$  was that the category  $\mathbf{Z}$  had a certain property  $C_n$ .

### 0.2. The current paper.

Our main results in this paper are the following.

- (i) We show that for a zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  between categories in which  $f$  has property  $B_n$  (0.1(ii)) (and in particular if  $\mathbf{Z}$  has property  $C_n$ ) (0.1(iii)), its *homotopy pullback* admits a description similar to the one mentioned in 0.1 namely as a *n-arrow pullback category*  $(f\mathbf{X} \downarrow_n g\mathbf{Y})$ . Moreover its *pullback* comes with a monomorphism into the homotopy pullback and hence is itself a homotopy pullback if the monomorphism is a weak equivalence.
- (ii) We then deduce from this similar results for zigzags in the categories of *relative categories* and *k-relative categories* ( $k > 1$ ) and do this not only with respect to their *Reedy structure*, but also with respect to their *Rezk structure* which turns them into models for the theories of  $(\infty, 1)$ - and  $(\infty, k)$ -categories respectively.
- (iii) We also note that a sufficient condition for a category, a relative category or a *k*-relative category to have *property*  $C_3$  is that it admits what we will call a *strict 3-arrow calculus*.

- (iv) Our main tool for proving all this consists of the *sharp maps* of Hopkins and Rezk [R2] which, because of their fibration-like properties we prefer to call *fibrillations*. They are the dual of Hopkins' *flat maps* which have similar cofibration-like properties and which we therefore call *cofibrillations*. These cofibrillations do not play any role in the current paper, except for a surprise appearance in 2.8(iii)'.

**0.3. The genesis of the current paper.** The original version of this paper consisted of only the results mentioned in 0.2(i) and the corresponding part of 0.2(iii). That was exactly what we needed in [BK3] where it enabled us to reduce the proof that certain pullbacks were homotopy pullbacks to a rather straightforward calculation. However our proof of these results was rather ad hoc and not very satisfactory.

Fortunately two things happened.

- (i) We discovered a manuscript of Charles Rezk [R2] in which he studied the *sharp maps* of Mike Hopkins. These maps seemed to be exactly what we needed. Just like fibrations, they could be used in a *right proper* model category to obtain pullbacks which were homotopy pullbacks (which by the way made us call them by the more suggestive name of *fibrillations*).
- (ii) Moreover when subsequently we took a closer look at the lemma of Quillen [Q, §1] which started it all and which he used to prove his Theorem B, we noted that this lemma was essentially just an elegant way of constructing fibrillations, some of which were exactly the ones we needed.

Combining all this with some simple properties of fibrillations we then not only obtained a much better proof of the above mentioned result 0.2(i), but also realized that with relatively little effort this proof could be extended to a proof of similar results for *relative categories* and the *k-relative categories* of [BK2], which as result the current manuscript.

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## CONTENTS

0. Introduction	1
1. An overview	3
<b>Part I. Categorical Preliminaries</b>	<b>6</b>
2. Relative categories	6
3. The categories and functors involved	8
4. Abstract nerve functors	12
<b>Part II. Homotopy pullback and potential homotopy pullback</b>	<b>14</b>
5. Homotopy pullbacks	14
6. Potential homotopy pullbacks	17
7. Properties $B_n$ and $C_n$	19
<b>Part III. The main results and their proofs</b>	<b>22</b>
8. The main results	22

9. Hopkins-Rezk fibrillations	24
10. A proof of Theorem 8.2	27
References	28

## 1. AN OVERVIEW

The paper consists of three parts, each of which consists of three sections.

### Part I: Categorical preliminaries

In sections 2, 3 and 4 we discuss the various models and relative categories involved and some of the relative functors between them.

#### 1.1. The categories involved.

- (i) The categories  $\mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 0$ ) of small  $k$ -relative categories of [BK2].  
 For  $k = 1$  this is just the category  $\mathbf{RelCat}$  of small relative categories of [BK1].  
 For  $k = 0$  this is the category of small maximal relative categories (i.e. in which all maps are weak equivalences). As  $\mathbf{Rel}^0 \mathbf{Cat}$  thus is isomorphic, although not equal, to the category  $\mathbf{Cat}$  of small categories, we will often denote  $\mathbf{Rel}^0 \mathbf{Cat}$  by  $\widehat{\mathbf{Cat}}$ .
- (ii) The categories  $s^k \widehat{\mathbf{Cat}}$  and  $s^k \mathbf{S}$  ( $k \geq 0$ ) which are respectively the categories of  $k$ -simplicial diagrams in  $\widehat{\mathbf{Cat}}$  and the categories of small  $k$ -simplicial spaces, i.e.  $(k+1)$ -simplicial sets.

#### 1.2. The functors involved.

- (i) The  $k$ -simplicial nerve functors  $s^k N: \mathbf{Rel}^k \mathbf{Cat} \rightarrow s^k \mathbf{S}$  ( $k \geq 0$ ) of [BK2], which for  $k = 0$  is just the classical nerve functor  $N$ .
- (ii) The functors  $N_*: s^k \widehat{\mathbf{Cat}} \rightarrow s^k \mathbf{S}$  ( $k \geq 0$ ) induced by  $N$ .
- (iii) The (unique) functors  $w_*: \mathbf{Rel}^k \mathbf{Cat} \rightarrow s^k \widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) such that  $N_* w_* = s^k N$ .

#### 1.3. The model and relative structures.

- (i) We endow  $\mathbf{S} = s^0 \mathbf{S}$  with the classical model structure and  $\widehat{\mathbf{Cat}}$  with the Quillen equivalent Thomason structure [T2].
- (ii) We endow  $s^k \mathbf{S}$  and  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 1$ ) with the resulting Reedy model structure.
- (iii) We endow  $\mathbf{RelCat}$  and  $\mathbf{Rel}^k \mathbf{Cat}$  ( $k > 1$ ) respectively with the Quillen equivalent Reedy model structure which [BK1] is lifted from the Reedy structure on  $s\mathbf{S}$  and the homotopy equivalent relative Reedy structure which [BK2] is lifted from the Reedy structure on  $s^k \mathbf{S}$ .
- (iv) In the application of our main result we will consider the categories  $\mathbf{Rel}^k \mathbf{Cat}$  and  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 1$ ) as models for the theory of  $(\infty, k)$ -categories by endowing them with a Rezk structure which has more weak equivalences than the Reedy structure and we will denote those categories so endowed by  $\mathbf{LRel}^k \mathbf{Cat}$  and  $\mathbf{L}s^k \widehat{\mathbf{Cat}}$ .

**1.4. Properties of the functors involved.** The functors considered in 1.2 above all have the same nice properties as the classical nerve functors (and we will therefore refer to them as *abstract nerve functors*).

In particular they all are *relative functors* which are *homotopy equivalences*.

## Part II: Homotopy pullbacks and potential homotopy pullbacks

1.5. In section 5 we will define homotopy pullbacks of zigzags in a model category in a more general fashion than is usually done, but that enables us to extend the definition to certain saturated relative categories.

- (i) In a model category we define a homotopy pullback of a zigzag as *any* object which is weakly equivalent to its image under a “*homotopically correct*” *homotopy limit functor*.
- (ii) In a saturated relative category we then define a homotopy pullback of a zigzag of a *any* object which is weakly equivalent to its image under what we will call a *weak homotopy limit functor* which is a functor which has only some of the properties of the above (i) homotopy limit functors.
- (iii) Our main result then is a *Global equivalence lemma* which states that
  - if  $f: \mathbf{C} \rightarrow \mathbf{D}$  is a homotopy equivalence between saturated relative categories, then  $\mathbf{C}$  admits weak homotopy limit functors iff  $\mathbf{D}$  does, and in that case  $f$  preserves homotopy pullbacks.
- (iv) This lemma not only
  - implies that any two weak homotopy limit functors on the same category yield the same notion of homotopy pullbacks of zigzags, and
  - enables us, in view of the fact that model categories admit such weak homotopy limit functors, to prove their existence elsewhere.
 but also plays a role in the proof of our main result in section 10.

1.6. In section 6 we discuss, for a zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $\mathbf{Rel}^k \mathbf{Cat}$  or  $\widehat{s^k \mathbf{Cat}}$  ( $k \geq 0$ ) the *n-arrow pullback object*  $(f\mathbf{X} \downarrow_n g\mathbf{Y})$  which was mentioned in 0.2 above.

1.7. In section 7 we recall the notions of *property  $B_n$*  and *property  $C_n$*  for  $\widehat{\mathbf{Cat}}$  which were mentioned in 0.1 above and then extend them to the categories  $\mathbf{Rel}^k \mathbf{Cat}$  and  $\widehat{s^k \mathbf{Cat}}$  for  $k \geq 1$ .

## Part III: The main results and their proof

1.8. In section 8 we formulate our main results, Theorems 8.2, 8.4, 8.6 and 8.7.

We also prove Theorem 8.4 and, assuming Theorem 8.2, Theorems 8.6 and 8.7 as well, but defer the proof of Theorem 8.2 itself until section 10.

- (i) Theorem 8.2 states that, for a zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $\mathbf{Rel}^k \mathbf{Cat}$  or  $\widehat{s^k \mathbf{Cat}}$  ( $k \geq 0$ ), for which the map  $f: \mathbf{X} \rightarrow \mathbf{Z}$  has property  $B_n$  or the object  $\mathbf{Z}$  has property  $C_n$  (1.7), the *n-arrow pullback object*  $(f\mathbf{X} \downarrow_n g\mathbf{Y})$  (1.6) is actually a *homotopy pullback* (1.5) of that zigzag.
- (ii) Theorem 8.2 states that a sufficient condition on an object  $\mathbf{Z} \in \mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 0$ ) in order that it has property  $C_3$  is that  $\mathbf{Z}$  admits what we will call a *strict 3-arrow calculus*.

- (iii) If  $k = 1$ , then this (ii) is in particular the case if  $\mathbf{Z}$  is a *partial model category*, i.e. a relative category which has the *two out of six* property and admits a *3-arrow calculus*.

These partial model categories have the property that [BK3], if  $\mathbf{RelCat}$  is endowed with the *Rezk* structure (1.3(iv)), then

- every partial model category is Reedy equivalent to a fibrant object in  $\mathbf{LRelCat}$  (1.3(iv)), and
- every fibrant object in  $\mathbf{LRelCat}$  is Reedy equivalent to a partial model category.

Consequently one has the following result for  $(\infty, 1)$ -categories.

- (iv) Theorem 8.6 that states that, for every zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $\mathbf{RelCat}$  between partial model categories, the *3-arrow pullback object*  $(f\mathbf{X} \downarrow_3 g\mathbf{Y})$  is a *homotopy pullback* of this zigzag not only in  $\mathbf{RelCat}$ , but also in  $\mathbf{LRelCat}$ , i.e. in  $(\infty, 1)$ -categories.
- (v) Finally in Theorem 8.7 we state a similar result for  $\mathbf{LRel}^k\mathbf{Cat}$  ( $k > 1$ ), i.e. for  $(\infty, k)$ -categories, which however is much weaker than that Theorem 8.6, as we have no model structure on  $\mathbf{Rel}^k\mathbf{Cat}$  for  $k > 1$  nor an analog for partial model categories.

1.9. In section 9 we review the *fibrillations* of Hopkins and Rezk which can be defined in any *relative category with pullbacks* and describe the following three lemmas which we need in section 10 to prove Theorem 8.2.

- (i) A *Quillen fibrillation lemma* which is a reformulation and a slight strengthening of the lemma that Quillen used to prove his Theorem B, and which enables us to obtain the needed fibrillations in  $\widehat{\mathbf{Cat}}$ .

If, for an object  $\mathbf{D} \in \widehat{\mathbf{Cat}}$  and a not necessarily relative functor  $F: \mathbf{D} \rightarrow \widehat{\mathbf{Cat}}$ ,  $\mathbf{Gr} F \in \widehat{\mathbf{Cat}}$  denotes its Grothendieck construction and  $\pi: \mathbf{Gr} F \rightarrow \mathbf{D} \in \widehat{\mathbf{Cat}}$  denotes the associated projection functor, then this lemma states that

- the projection functor  $\pi: \mathbf{Gr} F \rightarrow \mathbf{D} \in \widehat{\mathbf{Cat}}$  is a *fibrillation* iff the functor  $F: \mathbf{D} \rightarrow \widehat{\mathbf{Cat}}$  is a *relative* functor, i.e. sends every map of  $\mathbf{D}$  to a weak equivalence in  $\widehat{\mathbf{Cat}}$ .
- (ii) A *fibrillation lifting lemma* which enables us to obtain from the fibrillation in  $\widehat{\mathbf{Cat}}$  obtained in (i) above corresponding fibrillations in the categories  $s^k\widehat{\mathbf{Cat}}$  for  $k \geq 1$ .
- (iii) A *Hopkins-Rezk fibrillation lemma* which enables us to use the fibrillations of (ii) above to obtain the desired homotopy pullback in the categories  $s^k\mathbf{Cat}$  ( $k \geq 0$ ), i.e. it states that
- for every pullback square in a right proper model category

$$\begin{array}{ccc} \mathbf{U} & \longrightarrow & \mathbf{Y} \\ \downarrow & & \downarrow \\ \mathbf{V} & \longrightarrow & \mathbf{Z} \end{array}$$

in which the map  $\mathbf{C} \rightarrow \mathbf{Z}$  is a fibrillation, the object  $\mathbf{U}$  is a homotopy pullback of the zigzag  $\mathbf{V} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y}$ .

1.10. In section 10 we finally prove Theorem 8.2. We prove this first for the categories  $s^k\widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) by using the above (1.9) *three lemmas* and then lift these

results to the categories  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k \geq 1$ ) by means of the *Global equivalence lemma* which was mentioned in 1.5(iii) above.

Actually for  $\mathbf{RelCat}$  we did not have to use this fourth lemma as we could have obtained this also using only the first three.

## Part I. Categorical Preliminaries

### 2. RELATIVE CATEGORIES

2.1. **Summary.** We start with a brief review of

- (i) *relative categories* and *relative functors* between them,
- (ii) *saturated*, *maximal* and *minimal* relative categories,
- (iii) a *homotopy relation* between relative functors,
- (iv) *strict 3-arrow calculi* on relative categories,

and

- (v) the associated *simplicial nerve functor* to simplicial spaces, i.e. bisimplicial sets.

#### Relative Categories

2.2. **Definition.**

- (i) A **relative category** is a pair  $(\mathcal{C}, w\mathcal{C})$  (usually denoted by just  $\mathcal{C}$ ) consisting of an **underlying category**  $\mathcal{C}$  (sometimes denoted by  $\text{und } \mathcal{C}$ ) and a subcategory  $w\mathcal{C} \subset \mathcal{C}$  which is *only* required to contain *all* the objects of  $\mathcal{C}$  (and hence their identity maps).
- (ii) The maps of  $w\mathcal{C}$  are called **weak equivalences** and two objects of  $\mathcal{C}$  are called **weakly equivalent** if they can be connected by a finite zigzag of weak equivalences.
- (iii) By a **functor**  $f: \mathcal{C} \rightarrow \mathcal{D}$  between two relative categories we will mean just a functor  $f: \text{und } \mathcal{C} \rightarrow \text{und } \mathcal{D}$  (i), and
- (iv) such a functor will be called a **relative functor** if it preserves weak equivalences, i.e. if  $f(w\mathcal{C}) \subset w\mathcal{D}$ .

#### Saturated, maximal and minimal relative categories

2.3. **Definition.**

- (i) A relative category  $\mathcal{C}$  will be called **saturated** if a map in  $\mathcal{C}$  is a weak equivalence iff its image in  $\text{Ho } \mathcal{C}$  (i.e. the category obtained from  $\mathcal{C}$  by formally inverting all the weak equivalences) is an isomorphism.

This is in particular the case if

- (ii)  $\mathcal{C}$  is a **maximal** relative category, i.e. *all* maps of  $\mathcal{C}$  are weak equivalences

or if

- (iii)  $\mathcal{C}$  is a **minimal** relative category, i.e. the identity maps are the *only* weak equivalences.

#### A homotopy relation between relative functors

**2.4. Definition.** Let  $\mathbf{1}^w$  denote the maximal relative category (Df. 2.3(ii)) which consists of two objects 0 and 1 and one map  $0 \rightarrow 1$  between them. Then  $\mathbf{1}^w$  gives rise to the following notions.

- (i) Given two relative functors  $f, g: \mathbf{C} \rightarrow \mathbf{D}$  between relative categories, a **strict homotopy**  $h: f \rightarrow g$  between them will be a natural weak equivalence, i.e. a map

$$h: \mathbf{C} \times \mathbf{1}^w \longrightarrow \mathbf{D}$$

such that  $h(c, 0) = fc$  and  $h(c, 1) = gc$  for every object or map  $c \in \mathbf{C}$ .

- (ii) Two relative functors  $\mathbf{C} \rightarrow \mathbf{D}$  then are called **homotopic** if they can be connected by a finite zigzag of strict homotopies, and
- (iii) a functor  $f: \mathbf{C} \rightarrow \mathbf{D}$  is called a **homotopy equivalence** if  $f$  is a relative functor and if there exists a relative functor  $f': \mathbf{D} \rightarrow \mathbf{C}$  (called a **homotopy inverse** of  $f$ ) such that the compositions  $f'f$  and  $ff'$  are homotopic to  $1_{\mathbf{C}}$  and  $1_{\mathbf{D}}$  respectively.

### 3-arrow calculi

**2.5. Definition.** A relative category  $\mathbf{C}$  is said to admit a **3-arrow calculus** if there exist subcategories  $\mathbf{U}$  and  $\mathbf{V} \subset w\mathbf{C}$  which behave like the categories of the *trivial cofibrations* and *trivial fibrations* in a model category in the sense that

- (i) for every map  $u \in \mathbf{U}$ , its pushouts in  $\mathbf{C}$  exist and are again in  $\mathbf{U}$ ,
- (ii) for every map  $v \in \mathbf{V}$ , it's pullbacks in  $\mathbf{C}$  exist and are again in  $\mathbf{C}$ , and
- (iii) the maps  $w \in w\mathbf{C}$  admit a *functorial* factorization  $w = vu$  with  $u \in \mathbf{U}$  and  $v \in \mathbf{V}$ .

**2.6. Remark.** It should be noted that the conditions 2.5(i) and (ii) are stronger than the ones in [DK, 8.2] and [DHKS, 36.1]. However we prefer them as they are easier to use and are likely to be automatically satisfied.

**2.7. Remark.** In [DK, 8.2] and [DHKS, 36.1] 3-arrow calculi were defined in the presence of the *two out of three* and the *two out of six* properties respectively with the result that a *3-arrow calculus on  $(\mathbf{C}, w\mathbf{C})$  automatically restricted to a 3-arrow calculus on  $(w\mathbf{C}, w\mathbf{C})$* .

As we do not want to assume the presence of the two out of three or the two out of six property, however, we define a notion of *strict 3-arrow calculi* as follows.

### Strict 3-arrow calculi

**2.8. Definition.** A **strict 3-arrow calculus** on a relative category  $(\mathbf{C}, w\mathbf{C})$  will be a 3-arrow calculus (Df. 2.5) which restricts to a 3-arrow calculus on  $(w\mathbf{C}, w\mathbf{C})$ .

Another way of saying this by adding in Df. 2.5 to the conditions (i) and (ii) the conditions that

- (i)' every pushout of a map in  $w\mathbf{C}$  along a map in  $\mathbf{U}$  is again in  $w\mathbf{C}$ , and
- (ii)' every pullback of a map in  $w\mathbf{C}$  along a map in  $\mathbf{V}$  is again in  $w\mathbf{C}$ .

With other words the maps in  $\mathbf{U}$  and  $\mathbf{V}$  behave like trivial cofibrations and trivial fibrations in a *proper* model category.

A more compact way of saying all this is that

- (iii)' a strict 3-arrow calculus on a relative category is a functorial factorization of its weak equivalences into a *trivial cofibrillation* followed by a *trivial fibrillation*

where fibrillations are as defined in Df. 9.2 below and cofibrillations are defined dually.

### The simplicial nerve functor

**2.9. Definition.** Let  $\mathbf{RelCat}$  denote the category of (small) *relative categories* (Df. 2.2) and let  $\mathbf{sS}$  denote the category of **simplicial spaces**, i.e. *bisimplicial sets*.

Furthermore, for every integer  $p \geq 0$ , let  $\mathbf{p}$  denote the  $p$ -arrow category

$$0 \longrightarrow \cdots \longrightarrow p$$

and let  $\mathbf{p}^v$  and  $\mathbf{p}^w$  denote respectively the *minimal* and *maximal* relative categories (Df. 1.3(iii) and (ii)) which have  $\mathbf{p}$  as their underlying category.

The **simplicial nerve functor** then is the functor

$$\mathbf{sN}: \mathbf{RelCat} \longrightarrow \mathbf{sS}$$

which sends each object  $\mathbf{C} \in \mathbf{RelCat}$  to the bisimplicial set which in bidimension  $(p, q)$  consists of the maps

$$\mathbf{p}^v \times \mathbf{q}^w \longrightarrow \mathbf{C} \in \mathbf{RelCat}$$

### 3. THE CATEGORIES AND FUNCTORS INVOLVED

**3.1. Summary.** We now describe the categories and functors which we need in the paper and note that

- (i) these categories come with obvious notions of *strict homotopies* and *homotopy equivalences* (as in Df. 2.6),
- (ii) these functors *preserve* these strict homotopies and homotopy equivalences, and
- (iii) these functors all have *left adjoints* which are *left inverses*.

In section 4 we then endow these categories with model or relative structures which

- (iv) are compatible with these homotopy equivalences in the sense that *every homotopy equivalence is a weak equivalence*, and
- (v) turn the functors between these categories into relative functors which are *homotopy equivalences*.

### $k$ -relative categories

**3.2. Definition.** We extend the definition of  $k$ -relative categories for  $k \geq 1$  of [BK2, 3.3] to include also the case  $k = 0$  as follows.

A  $k$ -**relative category** ( $k > 0$ ) will be a  $(k+2)$ -tuple

$$\mathbf{C} = (a\mathbf{C}, v_1\mathbf{C}, \dots, v_k\mathbf{C}, w\mathbf{C})$$

consisting of an **ambient category**  $a\mathbf{C}$  and subcategories

$$v_1\mathbf{C}, \dots, v_k\mathbf{C} \text{ and } w\mathbf{C} \subset a\mathbf{C}$$



that each contain all the objects of  $a\mathbf{C}$  and form a commutative diagram with  $2k$  arrows of the form

$$\begin{array}{ccc} & w\mathbf{C} & \\ \swarrow & & \searrow \\ v_1\mathbf{C} & \cdots & v_k\mathbf{C} \\ \searrow & & \swarrow \\ & a\mathbf{C} & \end{array}$$

and where  $a\mathbf{C}$  is subject to the conditions that

- (i) every map in  $a\mathbf{C}$  is a finite composition of maps in the  $v_i\mathbf{C}$  ( $1 \leq i \leq k$ ), and
- (ii) every relation in  $a\mathbf{C}$  is the consequence of the commutativity in  $a\mathbf{C}$  of those sequences in  $a\mathbf{C}$  which are of the form

$$\begin{array}{ccc} \cdot & \xrightarrow{x_1} & \cdot \\ y_1 \downarrow & & \downarrow y_2 \\ \cdot & \xrightarrow{x_2} & \cdot \end{array}$$

in which  $x_1, x_2 \in v_i\mathbf{C}$  and  $y_1, y_2 \in v_j\mathbf{C}$  (where  $i$  and  $j$  are not necessarily distinct).

The maps of  $w\mathbf{C}$  are called **weak equivalences** and a **relative functor**  $f: \mathbf{C} \rightarrow \mathbf{D}$  between two  $k$ -relative categories will be a functor  $f: a\mathbf{C} \rightarrow a\mathbf{D}$  such that

$$f(w\mathbf{C}) \subset w\mathbf{D} \quad \text{and} \quad f(v_i\mathbf{C}) \subset v_i\mathbf{D} \quad \text{for all } i \leq k.$$

**3.3. Remark.** If  $\mathbf{C}$  is a 1-relative category, then  $a\mathbf{C} = v_1\mathbf{C}$  and hence

- 1-relative categories are the same as relative categories (Df. 2.2).

**3.4. Remark.** If  $\mathbf{C}$  is a 0-relative category, then  $a\mathbf{C} = w\mathbf{C}$  and hence

- 0-relative categories are the same as maximal relative categories (Df. 2.3(ii)).

### Strict 3-arrow calculi and $k$ -relative categories

**3.5. Definition.** A **strict 3-arrow calculus** on a  $k$ -relative category  $\mathbf{C}$  is a 3-arrow calculus (Df. 2.5) on  $(a\mathbf{C}, w\mathbf{C})$  which restricts to a 3-arrow calculus on

$$(w\mathbf{C}, w\mathbf{C}) \quad \text{and} \quad (v_i\mathbf{C}, w\mathbf{C}) \quad \text{for all } i \leq k.$$

### The $k$ -simplicial nerve functor

**3.6. Definition.** Let  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k \geq 0$ ) denote the category of (small)  $k$ -relative categories (Df. 3.2) and let  $s^k\mathbf{S}$  denote the category of (small)  $k$ -simplicial spaces, i.e.  $(k+1)$ -simplicial sets.

Furthermore, for every integer  $p \geq 0$  let  $|\mathbf{p}| \subset \mathbf{p}$  (Df. 2.9) denote the subcategory which consists of the objects and their identity maps only and let

$$\mathbf{p}^w \text{ and } \mathbf{p}^{v_i} \text{ } (1 \leq i \leq k) \in \mathbf{Rel}^k\mathbf{Cat}$$

be such that

$$ap^w = v_i p^w = wp^w = p \quad (1 \leq i \leq k), \text{ and} \\ ap^{v_i} = v_i p^{v_i} = p \quad \text{and} \quad v_j p^{v_i} = wp^{v_i} = |p| \quad \text{for } j \neq i.$$

The  $k$ -simplicial nerve functor then is the functor

$$s^k N: \mathbf{Rel}^k \mathbf{Cat} \longrightarrow s^k \mathbf{S}$$

which each object  $C \in \mathbf{Rel}^k \mathbf{Cat}$  to the  $(k+1)$ -simplicial set which in multidimension  $(p_k, \dots, p_1, q)$  consists of the maps

$$p_k^{v_k} \times \dots \times p_1^{v_1} \times q^w \longrightarrow C \in \mathbf{Rel}^k \mathbf{Cat}$$

**3.7. Remark.** If  $k = 1$ , then (Rk. 3.3)  $\mathbf{Rel}^k \mathbf{Cat} = \mathbf{RelCat}$  (Df. 2.9) and hence in that case Df. 3.6 agrees with Df. 2.9.

**3.8. Notation.** If  $k = 0$ , then

- (i) the category  $\mathbf{Rel}^0 \mathbf{Cat}$  is, in view of Rk. 3.4, isomorphic, although not identical, with the category  $\mathbf{Cat}$  of (small) categories and we will therefore
  - often denote  $\mathbf{Rel}^0 \mathbf{Cat}$  by  $\widehat{\mathbf{Cat}}$ .
- (ii)  $s^0 \mathbf{S} = \mathbf{S}$ , the category of (small) simplicial sets, and
- (iii) the functor  $s^0 N: \mathbf{Rel}^0 \mathbf{Cat} \rightarrow s^0 \mathbf{S}$  is just the **classical nerve functor** which we will therefore denote by

$$N: \widehat{\mathbf{Cat}} \longrightarrow \mathbf{S}$$

Closely related to the  $k$ -simplicial nerve functor are the following two functors.

### The levelwise nerve functor

**3.9. Definition.** Let  $k \geq 0$ . Then the **levelwise nerve functor**

$$N_*: s^k \widehat{\mathbf{Cat}} \longrightarrow s^k \mathbf{S}$$

will be the functor in which  $s^k \widehat{\mathbf{Cat}}$  denotes the category of  $k$ -simplicial diagrams in  $\widehat{\mathbf{Cat}}$  and  $N_*$  is dimensionwise application of the classical nerve functor  $N$  (Nt. 3.3).

### The higher equivalence functor

**3.10. Definition.** For every integer  $k \geq 0$  we denote by

$$w_*: \mathbf{Rel}^k \mathbf{Cat} \longrightarrow s^k \widehat{\mathbf{Cat}}$$

the **higher equivalence functor** which, for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$  sends each object  $C \in \mathbf{Rel}^k \mathbf{Cat}$  to the object  $w_{p_*} C \in \widehat{\mathbf{Cat}}$  which has as its objects the maps

$$p_k^{v_k} \times \dots \times p_1^{v_1} \longrightarrow C \in \mathbf{Rel}^k \mathbf{Cat}$$

and as maps the natural weak equivalences between them.

The above functors have the following properties.

**3.11. Proposition.** *The following diagram commutes*

$$\begin{array}{ccc}
 \mathbf{Rel}^k \mathbf{Cat} & \xrightarrow{s^k N} & s^k \mathbf{S} \\
 & \searrow w_* & \nearrow N_x \\
 & s^k \widehat{\mathbf{Cat}} & 
 \end{array} \quad k \geq 0$$

*Proof.* This follows directly from the definitions.  $\square$

**3.12. Proposition.** *For every integer  $k \geq 0$  the functors*

- (i)  $N_* : s^k \widehat{\mathbf{Cat}} \rightarrow s^k \mathbf{S}$ ,
- (ii)  $s^k N : \mathbf{Rel}^k \mathbf{Cat} \rightarrow s^k \mathbf{S}$ , and
- (iii)  $w_* : \mathbf{Rel}^k \mathbf{Cat} \rightarrow s^k \widehat{\mathbf{Cat}}$

*have a left adjoint which is a left inverse.*

*Proof.*

- (i) It clearly suffices to prove this for the functor  $N$ . But this was already done long ago in [GZ, II, 4.7].
- (ii) This was shown in [BK2, 4.4].
- (iii) This follows from Pr. 3.11 above and the following result.  $\square$

**3.13. Proposition.** *If, for two functors  $g_1 : \mathbf{A} \rightarrow \mathbf{B}$  and  $g_2 : \mathbf{B} \rightarrow \mathbf{C}$ , both  $g_2$  and  $g_2 g_1$  have left adjoints which are left inverses, then so does  $g_1$ .*

*Proof.* Consider the diagram

$$\begin{array}{ccccc}
 & & f_{12} & & \\
 & \swarrow & \overline{\phantom{f_{12}}} & \searrow & \\
 \mathbf{A} & \xleftarrow{f_1} & \mathbf{B} & \xleftarrow{f_2} & \mathbf{C} \\
 & \searrow g_1 & & \searrow g_2 & \\
 & & & & 
 \end{array}$$

in which  $f_2$  and  $f_{12}$  are left adjoints of  $g_2$  and  $g_2 g_1$  and

$$f_2 g_2 = 1, \quad f_{12} g_2 g_1 = 1 \quad \text{and} \quad f_1 = f_{12} g_2.$$

Then  $f_1 g_1 = f_{12} g_2 g_1 = 1$  and it thus remains to show that  $f_1$  is a left adjoint of  $g_1$  and this one does by noting that every pair of objects  $A \in \mathbf{A}$  and  $B \in \mathbf{B}$  gives rise to a natural sequence of identity maps and isomorphisms

$$\begin{array}{ccc}
 \mathbf{A}(f_1 B, A) & & \mathbf{B}(B, g_1 A) \\
 \parallel & & \parallel \\
 \mathbf{A}(f_{12} g_2 B, A) \approx \mathbf{C}(g_2 B, g_2 g_1 A) \approx \mathbf{B}(f_2 g_2 B, g_1 A) & & 
 \end{array} \quad \square$$

It remains to deal with the results that were promised in 3.1(i) and (ii).

### Homotopy relations on $\mathbf{Rel}^k \mathbf{Cat}$ , $s^k \mathbf{S}$ and $s^k \widehat{\mathbf{Cat}}$

**3.14. Definition.** One can define **strict homotopies** and **homotopy equivalences** in the categories  $\mathbf{Rel}^k \mathbf{Cat}$ ,  $s^k \mathbf{S}$  and  $s^k \widehat{\mathbf{Cat}}$  as in Df. 2.4 by choosing an “obvious analog” of  $\mathbf{1}^w$  as follows.

- (i) For  $\mathbf{Rel}^k \mathbf{Cat}$  this will be the object  $\mathbf{1}^w \in \mathbf{Rel}^k \mathbf{Cat}$  (Df. 3.6),
- (ii) for  $s^k \mathbf{S}$  this will be the standard multisimplex  $\Delta[0, \dots, 0, 1] \in s^k \mathbf{S}$ , and
- (iii) for  $s^k \widehat{\mathbf{Cat}}$  this will be the object  $\mathbf{1}_x^w \in s^k \widehat{\mathbf{Cat}}$  such that, for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$ ,  $\mathbf{1}_{p_*}^w = \mathbf{1}^w \in \widehat{\mathbf{Cat}}$ .

**3.15. Proposition.** *The functors  $s^k N$ ,  $w_*$  and  $N_*$*

- (i) *send strictly homotopic maps to strictly homotopic maps and hence*
- (ii) *send homotopy equivalences to homotopy equivalences.*

*Proof.*

- (i) The classical nerve functor  $N: \widehat{\mathbf{Cat}} \rightarrow \mathbf{S}$  has the property that ([La, §2] and [Le]), for every two maps  $f, g: \mathbf{A} \rightarrow \mathbf{B} \in \widehat{\mathbf{Cat}}$ 
  - $N$  sends every strict homotopy between  $f$  and  $g$  to a strict homotopy between  $Nf$  and  $Ng$ , and
  - conversely, every strict homotopy in  $\mathbf{S}$  between  $Nf$  and  $Ng$  is obtained in this fashion.
- (ii) This readily implies that  $N_*$  has the same properties.
- (iii) The proof for  $s^k N$  is just a higher dimensional version of the proof for  $k = 1$  in [BK1, 7.5].
- (iv) A proof for  $w_*$  then is obtained by combining Pr. 3.11 with (ii) and (iii) above.  $\square$

#### 4. ABSTRACT NERVE FUNCTORS

**4.1. Summary.** We now endow the categories  $\mathbf{Rel}^k \mathbf{Cat}$ ,  $s^k \mathbf{S}$  and  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) with a *relative* or *model* structure which will turn the functors  $s^k N$ ,  $w_*$  and  $N_*$  into what we will call *abstract nerve functors*, i.e. functors which, in addition to the properties that we mentioned in Pr. 3.12 and Pr. 3.15, have the property that

- (i) in the categories involved all *homotopy equivalences* are *weak equivalences*, and
- (ii) the functors between them are *homotopy equivalences*.

We do this in two different ways. In the first we endow the categories  $\mathbf{Rel}^k \mathbf{Cat}$ ,  $s^k \mathbf{S}$  and  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 1$ ) with *Reedy* structures. In the second we endow them with *Rezk* structures which have more weak equivalences and which turn these categories into models for the theory of  $(\infty, k)$ -categories.

##### Abstract nerve functors

**4.2. Definition.** A functor  $f: \mathbf{C} \rightarrow \mathbf{D}$  between saturated relative categories (Df. 2.3) will be called an **abstract nerve functor** if it has the following four properties:

- (i) The relative categories  $\mathbf{C}$  and  $\mathbf{D}$  come with *strict homotopies* for which the associated *homotopy equivalences* are *weak equivalences*.
- (ii) The functor  $f$  is a relative functor which is a *homotopy equivalence*.
- (iii) The functor  $f$  sends strictly homotopic maps to strictly homotopic maps and hence sends *homotopy equivalences* to *homotopy equivalences*.

- (iv) The functor  $f$  has a *left adjoint* which is a *left inverse*.

### The Reedy structures

4.3. **Definition.** We endow

- (i) the category  $\mathcal{S}$  with the usual model structure,
- (ii) the category  $\widehat{\mathbf{Cat}}$  with the *Quillen equivalent* Thomason structure [T2],
- (iii) the categories  $s^k\mathcal{S}$  and  $s^k\widehat{\mathbf{Cat}}$  ( $k \geq 1$ ) with the resulting Reedy model structures,
- (iv) the category  $\mathbf{RelCat}$  with the *Quillen equivalent* Reedy model structure of [BK1, 6.1] which was lifted from the Reedy structure on  $s\mathcal{S}$ , and
- (v) the categories  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k > 1$ ) with the *homotopy equivalent* relative Reedy structures of [BK2, 6.2] which were lifted from the Reedy structures on  $s^k\mathcal{S}$ .

4.4. **Proposition.** *If the categories  $\mathbf{simp}^k\mathcal{S}$ ,  $\mathbf{Rel}^k\mathbf{Cat}$  and  $s^k\widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) are endowed with the relative structures of 4.3, then the functors  $N_*$ ,  $s^k N$  and  $w_*$  are abstract nerve functors.*

*Proof.* In view of Pr. 3.12 and 3.15 it suffices to show that

- (i) the functors  $N_*$ ,  $s^k N$  ( $k \geq 1$ ) and  $w_*$  are homotopy equivalences, and
- (ii) the homotopy equivalences in  $s^k\mathcal{S}$ ,  $\mathbf{Rel}^k\mathbf{Cat}$  and  $s^k\widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) are weak equivalences.

To prove (i) we note:

- a) As Dana Latch [La] has shown that the functor  $N$  is a homotopy equivalence it readily follows that so are the functors  $N_*$ .
- b) That the functors  $s^k N$  are homotopy equivalences was shown in [BK2, 5.1].
- c) That the functors  $w_*$  are homotopy equivalences then follows a), b) and Pr. 3.11 and the observation that homotopy equivalences have the two out of three property.

To prove (ii) we note

- d) As the homotopy equivalences in  $\mathcal{S}$  are weak equivalences, it readily follows that so are the homotopy equivalences in  $s^k\mathcal{S}$ .
- e) That the homotopy equivalences in  $\mathbf{Rel}^k\mathbf{Cat}$  and  $s^k\widehat{\mathbf{Cat}}$  are also weak equivalences then follows from d) and the fact that the functors  $s^k N$  and  $N_*$  preserve homotopy equivalences and reflect weak equivalences.  $\square$

### The Rezk structures

4.5. **Definition.** In [R1] Charles Rezk constructed a left Bousfield localization of the Reedy structure on  $s\mathcal{S}$  and showed it to be a model for the theory of  $(\infty, 1)$ -categories.

Furthermore it was noted in [B] (and a proof thereof can be found in [Lu, §1]) that iteration of Rezk's construction yields for every integer  $k > 1$ , a left Bousfield localization of the Reedy structure on  $s^k\mathcal{S}$  which is a model for the theory of  $(\infty, k)$ -categories.

We therefore will denote

- (i) by  $\mathbf{Ls}^k\mathcal{S}$  ( $k \geq 1$ ) the category  $s^k\mathcal{S}$  with this (iterated) Rezk structure,

- (ii) by  $\text{Ls}^k\widehat{\mathbf{Cat}}$  ( $k \geq 1$ ) the induced [H, 3.3.20] *Quillen equivalent* Rezk model structure, and
- (iii) by  $\mathbf{LRel}^k\mathbf{Cat}$  the *Quillen or homotopy equivalent* Rezk structure lifted [BK2, 4.2] from the Rezk structure on  $s^k\mathbf{S}$  (or the Quillen equivalent Rezk structure on  $s^k\widehat{\mathbf{Cat}}$ ) which categories therefore are all models for the theory of  $(\infty, k)$ -categories.

**4.6. Proposition.** *If the categories  $s^k\mathbf{S}$ ,  $\mathbf{Rel}^k\mathbf{Cat}$  and  $s^k\widehat{\mathbf{Cat}}$  are endowed with the Rezk structures of 4.5, then the functors  $s^k\mathbf{N}$ ,  $w_*$  and  $\mathbf{N}_*$  are abstract nerve functors.*

*Proof.* This follows from Pr. 4.4 and the fact that the Rezk structures have more weak equivalences than the Reedy ones.  $\square$

## Part II. Homotopy pullback and potential homotopy pullback

### 5. HOMOTOPY PULLBACKS

**5.1. Summary.** As we are concerned not only with homotopy pullbacks in the *model categories*  $\mathbf{RelCat}$  and  $s^k\widehat{\mathbf{Cat}}$  ( $k \geq 0$ ), but also in the *saturated relative categories*  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k > 1$ ) on which we do *not* have a model structure, we will define homotopy pullback in a more general fashion than is usually done.

- (i) In a model category we define a homotopy pullback of a zigzag as *any object* which is weakly equivalent to its image under a “*homotopically correct*” *homotopy limit functor*.
- (ii) In a saturated relative category we then define a homotopy pullback of a zigzag as *any object* weakly equivalent to its image under what we will call a *weak homotopy limit functor* which is a functor which has only some of the properties of the above (i) homotopy limit functors.
- (iii) Our main result then is a *global equivalence lemma* which states that, if  $f: \mathbf{C} \rightarrow \mathbf{D}$  is a homotopy equivalence between saturated relative categories, then  $\mathbf{C}$  admits weak homotopy limit functors iff  $\mathbf{D}$  does, and in that case  $f$  preserves homotopy pullbacks.

In view of Df. 4.2(ii) and Pr. 4.4 this result not only takes care of the notion of homotopy pullback in the categories  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k > 1$ ), but it enables us, in the proof of our main result in section 10, to lift our results from the model categories  $s^k\widehat{\mathbf{Cat}}$  ( $k > 1$ ) to the relative categories  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k > 1$ ).

**5.2. Remark.** The results of this section actually hold for homotopy limit functors (and dually homotopy colimit functors) on *arbitrary* diagram categories.

**5.3. Remark.** As our definition of homotopy pullbacks is much less rigid than the usual ones, it might be more correct to refer to them as *models for homotopy limits*.

### Homotopy pullbacks in model categories and their left Bousfield localizations

5.4. **Definition.** Let  $\mathbf{E}$  denote the 2-arrow category  $\cdot \rightarrow \cdot \leftarrow \cdot$ .

Given a model category  $\mathbf{M}$ , we then mean by a **homotopy  $\mathbf{E}$ -limit functor** on  $\mathbf{M}$  a “homotopically correct” homotopy limit functor

$$\mathrm{holim}^{\mathbf{E}}: \mathbf{M}^{\mathbf{E}} \longrightarrow \mathbf{M},$$

i.e. a functor which, as for instance in [DHKS, 20.1], sends every object of  $\mathbf{M}^{\mathbf{E}}$  to a *fibrant* object of  $\mathbf{M}$  and every (objectwise) weak equivalence in  $\mathbf{M}^{\mathbf{E}}$  to a *weak equivalence* in  $\mathbf{M}$ .

It has the following property.

5.5. **Proposition.** *The functor*

$$\mathrm{Ho} \mathrm{holim}^{\mathbf{E}}: \mathrm{Ho}(\mathbf{M}^{\mathbf{E}}) \longrightarrow \mathrm{Ho} \mathbf{M}$$

*is a right adjoint of the constant diagram functor  $\mathrm{Ho} \mathbf{M} \rightarrow \mathrm{Ho}(\mathbf{M}^{\mathbf{E}})$ .*

*Proof.* This is a special case of [DHKS, 20.2]. □

5.6. **Definition.** Given an object  $\mathbf{B} \in \mathbf{M}^{\mathbf{E}}$ , we will say that an object  $\mathbf{U} \in \mathbf{M}$  is a **homotopy pullback** of  $\mathbf{B}$ , if  $\mathbf{U}$  is weakly equivalent to  $\mathrm{holim}^{\mathbf{E}} \mathbf{B}$ .

### Quasi-fibrant objects in left Bousfield localizations of left proper model categories

5.7. **Definition.** Let  $\mathbf{M}$  be a model category and let  $\mathbf{LM}$  be a left Bousfield localization of  $\mathbf{M}$ , i.e. [H, 3.3.3] a model category with the *same* cofibrations but *more* weak equivalences.

If  $\mathbf{M}$  is left proper, then an object  $\mathbf{D} \in \mathbf{LM}$  will be called **quasi-fibrant** if

- (i)  $\mathbf{X}$  is weakly equivalent in  $\mathbf{M}$  to a fibrant object in  $\mathbf{LM}$

or equivalently [H, 3.4.6(1)]

- (ii) one (and hence every) fibrant approximation of  $\mathbf{X}$  in  $\mathbf{M}$  is fibrant in  $\mathbf{LM}$ .

5.8. **Proposition.** Let  $\mathbf{LM}$  be a left Bousfield localization of a left proper model category  $\mathbf{M}$ . Then, for every zigzag between quasi-fibrant objects, its homotopy pullbacks in  $\mathbf{M}$  are quasi-fibrant in  $\mathbf{LM}$  and also homotopy pullbacks of this zigzag in  $\mathbf{LM}$ .

*Proof.* This follows readily from [H, 3.4.6(1)] and [H, 19.6.5]. □

### Homotopy pullbacks in saturated relative categories

5.9. **Definition.** Give a saturated relative category  $\mathbf{R}$  (Df. 2.3(i)), a **weak homotopy  $\mathbf{E}$ -limit functor** will be a relative functor

$$\mathrm{wholim}^{\mathbf{E}}: \mathbf{R}^{\mathbf{E}} \longrightarrow \mathbf{R}$$

for which the induced functor

$$\mathrm{Ho} \mathrm{wholim}^{\mathbf{E}}: \mathrm{Ho}(\mathbf{R}^{\mathbf{E}}) \longrightarrow \mathrm{Ho} \mathbf{R}$$

is a right adjoint of the constant diagram functor  $\mathrm{Ho} c: \mathrm{Ho} \mathbf{R} \rightarrow \mathrm{Ho}(\mathbf{R}^{\mathbf{E}})$ .

5.10. **Definition.** Given an object  $\mathbf{B} \in \mathbf{R}^{\mathbf{E}}$ , we will say that an object  $\mathbf{U} \in \mathbf{R}$  is a **homotopy pullback** of  $\mathbf{B}$  if  $\mathbf{U}$  is weakly equivalent to  $\mathrm{wholim}^{\mathbf{E}} \mathbf{B}$ .

**5.11. Remark.** While for a model category any two homotopy limit functors (Df. 5.4) are naturally weakly equivalent, this need not be the case for these *weak* homotopy limit functors. However they still have the following *local equivalence* property which, because of our *non-functorial* definition of homotopy pullbacks, is all we will need.

**5.12. Proposition.** *The notion of homotopy pullbacks does not depend on the choice of weak homotopy  $E$ -limit functor as*

(i) *for any two such functors*

$$\mathrm{wholim}_1^E \quad \text{and} \quad \mathrm{wholim}_2^E : \mathbf{R}^E \longrightarrow \mathbf{R}$$

*the induced functors*

$$\mathrm{Ho} \mathrm{wholim}_1^E \quad \text{and} \quad \mathrm{Ho} \mathrm{wholim}_2^E : \mathrm{Ho}(\mathbf{R}^E) \longrightarrow \mathrm{Ho} \mathbf{R}$$

*are naturally isomorphic*

*which implies that*

(ii) *for every object  $B \in \mathbf{R}^E$ , the objects  $\mathrm{wholim}_1^E B$  and  $\mathrm{wholim}_2^E B$  are weakly equivalent.*

*Proof.* This follows readily from the uniqueness of adjoints and the saturation of  $\mathbf{R}$ .  $\square$

We end with another useful property.

### A global equivalence lemma

**5.13. Lemma.** *Let  $f : \mathbf{R}_1 \rightarrow \mathbf{R}_2$  be a homotopy equivalence (Df. 2.4(iii)) between saturated relative categories. Then*

(i) *there exist weak homotopy  $E$ -limit functors on  $\mathbf{R}_1$  iff they exist on  $\mathbf{R}_2$*

*in which case*

(ii) *an object  $U \in \mathbf{R}_1$  is a homotopy pullback of an object  $B \in \mathbf{R}_1^E$  iff the object  $fU \in \mathbf{R}_2$  is a homotopy pullback of the object  $fB \in \mathbf{R}_2^E$ .*

*Proof.* If  $\mathrm{wholim}_1^E : \mathbf{R}_1^E \rightarrow \mathbf{R}_1$  is a weak homotopy limit functor and  $g : \mathbf{R}_2 \rightarrow \mathbf{R}_1$  is a homotopy inverse (Df. 2.4(iii)) of  $f$ , then it suffices to show that the following composition is also a weak homotopy limit functor

$$\mathbf{R}_2 \xleftarrow{f} \mathbf{R}_1 \xleftarrow{\mathrm{wholim}_1^E} \mathbf{R}_1^E \xleftarrow{g^E} \mathbf{R}_2^E$$

To do this we successively note the following.

(i) The maps

$$\mathrm{Ho} f : \mathrm{Ho} \mathbf{R}_1 \longrightarrow \mathrm{Ho} \mathbf{R}_2 \quad \text{and} \quad \mathrm{Ho} g : \mathrm{Ho} \mathbf{R}_2 \longrightarrow \mathrm{Ho} \mathbf{R}_1$$

are inverse equivalences of categories and hence are both left and right adjoint.

(ii) This implies the existence of the sequence of adjunctions

$$\mathrm{Ho} \mathbf{R}_2 \xrightleftharpoons[\mathrm{Ho} f]{\mathrm{Ho} g} \mathrm{Ho} \mathbf{R}_1 \xrightleftharpoons[\mathrm{Ho} \mathrm{wholim}_1^E]{\mathrm{Ho} i_1} \mathrm{Ho}(\mathbf{R}_1^E) \xrightleftharpoons[\mathrm{Ho} g^E]{\mathrm{Ho} f^E} \mathrm{Ho}(\mathbf{R}_2^E)$$



(iii) The composition of the left adjoints equals the composition

$$\mathrm{Ho} \mathbf{R}_2 \xrightarrow{\mathrm{Ho} g} \mathrm{Ho} \mathbf{R}_1 \xrightarrow{\mathrm{Ho} f} \mathrm{Ho} \mathbf{R}_2 \xrightarrow{\mathrm{Ho} i_2} \mathrm{Ho}(\mathbf{R}_2^E)$$

(iv) As  $(\mathrm{Ho} f)(\mathrm{Ho} g)$  is naturally isomorphic to the identity of  $\mathrm{Ho} \mathbf{R}_2$ , the composition in (iii) is naturally isomorphic to

$$\mathrm{Ho} i_2: \mathrm{Ho} \mathbf{R}_2 \longrightarrow \mathrm{Ho}(\mathbf{R}_2^E)$$

which implies that this map is a left adjoint of the composition of the right adjoints in (iii).  $\square$

## 6. POTENTIAL HOMOTOPY PULLBACKS

**6.1. Summary.** For every integer  $n \geq 1$  we will in a functorial manner embed every zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in the categories  $\mathbf{Rel}^k \mathbf{Cat}$  and  $\widehat{\mathbf{s}^k \mathbf{Cat}}$  ( $k \geq 0$ ) (Df. 3.6 and 3.9) in a commutative diagram of the form

$$\begin{array}{ccccc} \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} & \xrightarrow{k} & (f \mathbf{X} \downarrow_n g \mathbf{Y}) & \xrightarrow{\pi} & \mathbf{Y} \\ \downarrow & & \downarrow & & \downarrow g \\ \mathbf{X} & \xrightarrow{h} & (f \mathbf{X} \downarrow_n \mathbf{Z}) & \xrightarrow{\pi} & \mathbf{Z} \end{array}$$

in which

- (i) the squares are pullback squares,
- (ii)  $h$  is a weak equivalence, and
- (iii) the object  $(f \mathbf{X} \downarrow_n g \mathbf{Y})$  is a *potential* homotopy pullback of this zigzag in the sense that, under suitable restrictions on the map  $f: \mathbf{X} \rightarrow \mathbf{Z}$  (which will be discussed in section 7), this object is indeed a homotopy pullback of this zigzag.

We start with an auxiliary construction.

### $n$ -arrow path objects in $\mathbf{Rel}^k \mathbf{Cat}$ ( $k \geq 0$ )

**6.2. Definition.** Given an integer  $n \geq 1$  and an object  $\mathbf{Z} \in \mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 0$ ) we denote by  $(\mathbf{Z} \downarrow_n \mathbf{Z}) \in \mathbf{Rel}^k \mathbf{Cat}$  the  $n$ -arrow path object which has

- (i) as objects the  $n$ -arrow zigzags

$$Z_n \cdots Z_2 \longleftarrow Z_1 \longrightarrow Z_0 \quad \text{in } w\mathbf{Z},$$

- (ii) as maps in  $w(\mathbf{Z} \downarrow_n \mathbf{Z})$  and  $v_i(\mathbf{Z} \downarrow_n \mathbf{Z})$  ( $1 \leq i \leq k$ ) the commutative diagrams of the form

$$\begin{array}{ccccccc} Z_n & \cdots & Z_2 & \longleftarrow & Z_1 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z'_n & \cdots & Z'_2 & \longleftarrow & Z'_1 & \longrightarrow & Z'_0 \end{array} \quad \text{in } a\mathbf{Z}$$

in which the vertical maps are in  $w\mathbf{Z}$  and  $v_i\mathbf{Z}$  respectively, and

- (iii) as maps in  $a(\mathbf{Z} \downarrow_n \mathbf{Z})$  those commutative diagrams as above which are finite compositions of maps in the  $v_i(\mathbf{Z} \downarrow_n \mathbf{Z})$  ( $1 \leq i \leq k$ ).

Furthermore

(iv) we denote by

$$\mathbf{Z} \xrightarrow{\pi_n} (\mathbf{Z} \downarrow_n \mathbf{Z}) \xrightarrow{\pi_0} \mathbf{Z} \quad \text{and} \quad \mathbf{Z} \xrightarrow{j} (\mathbf{Z} \downarrow_n \mathbf{Z})$$

the restrictions of  $(\mathbf{Z} \downarrow_n \mathbf{Z})$  to the first and last entries respectively and the map which sends each object of  $\mathbf{Z}$  to the alternating zigzag of its identity maps.

These maps have the following nice properties.

**6.3. Proposition.**

- (i)  $\pi_n j = \pi_0 j = 1$ ,
- (ii)  $j$  is a homotopy equivalence which has  $\pi_n$  and  $\pi_0$  as homotopy inverses and hence (Df. 4.3(iv) and (v) and Pr. 4.4)
- (iii) all three maps are weak equivalences in  $\mathbf{Rel}^k \mathbf{Cat}$ .

*Proof.* This is a straightforward computation.  $\square$

**$n$ -arrow fibers and pullback objects in  $\mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 0$ )**

**6.4. Definition.** Given an integer  $n \geq 1$  and a zigzag in  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $\mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 0$ )

- (i) we denote by  $(f\mathbf{X} \downarrow_n \mathbf{Z})$  the  $n$ -arrow fibers object which is defined by

$$(f\mathbf{X} \downarrow_n \mathbf{Z}) = \mathbf{X} \times_{\mathbf{Z}} (\mathbf{Z} \downarrow_n \mathbf{Z}) = \lim(\mathbf{X} \xrightarrow{f} \mathbf{Z} \xleftarrow{\pi_n} (\mathbf{Z} \downarrow_n \mathbf{Z}))$$

and note that it comes with a **projection map**

$$\pi: (f\mathbf{X} \downarrow_n \mathbf{Z}) \longrightarrow \mathbf{Z}$$

induced by the map  $\pi_0: (\mathbf{X} \downarrow_n \mathbf{Z}) \rightarrow \mathbf{Z}$ , and

- (ii) we denote by  $(f\mathbf{X} \downarrow_n g\mathbf{Y})$  the  $n$ -arrow pullback object which is defined by

$$(f\mathbf{X} \downarrow_n g\mathbf{Y}) = (f\mathbf{X} \downarrow_n \mathbf{Z}) \times_{\mathbf{Z}} \mathbf{Y} = \lim((f\mathbf{X} \downarrow_n \mathbf{Z}) \xrightarrow{\pi} \mathbf{Z} \xleftarrow{g} \mathbf{Y})$$

and note that it comes with a **projection map**

$$\pi: (f\mathbf{X} \downarrow_n g\mathbf{Y}) \longrightarrow \mathbf{Y}$$

obtained by “restriction to  $\mathbf{Y}$ ”.

**$n$ -arrow fibers and pullback objects in  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 0$ )**

**6.5. Definition.** As (Nt. 3.4)  $\mathbf{Rel}^0 \mathbf{Cat} = \widehat{\mathbf{Cat}}$ , the case  $s^0 \widehat{\mathbf{Cat}} = \widehat{\mathbf{Cat}}$  has already been taken care of in Df. 6.4.

Given an integer  $n \geq 1$  and a zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $s^k \widehat{\mathbf{Cat}}$  ( $k \leq 1$ ) we can therefore define the  $n$ -arrow fibers and pullback objects  $(f\mathbf{Z} \downarrow_n \mathbf{Z})$  and  $(f\mathbf{X} \downarrow_n g\mathbf{Y})$  and the associated **projection maps**

$$\pi: (f\mathbf{X} \downarrow_n \mathbf{Z}) \longrightarrow \mathbf{Z} \quad \text{and} \quad \pi: (f\mathbf{X} \downarrow_n g\mathbf{Y}) \longrightarrow \mathbf{Y}$$

by the requirement that, for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$

$$((f\mathbf{X} \downarrow_n \mathbf{Z}) \xrightarrow{\pi} \mathbf{Z})_{p_*} = ((f_{p_*} \mathbf{X}_{p_*} \downarrow_n \mathbf{Z}_{p_*}) \xrightarrow{\pi} \mathbf{Z}_{p_*}) \in \widehat{\mathbf{Cat}}, \quad \text{and}$$

$$((f\mathbf{X} \downarrow_n g\mathbf{Y}) \xrightarrow{\pi} \mathbf{Y})_{p_*} = ((f_{p_*} \mathbf{X}_{p_*} \downarrow_n g_{p_*} \mathbf{Y}_{p_*}) \xrightarrow{\pi} \mathbf{Y}_{p_*}) \in \widehat{\mathbf{Cat}}$$

The definitions 6.4 and 6.5 are closely related as follows.

**6.6. Proposition.** *For every integer  $n \geq 1$  and zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $\mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 1$ ) (Df. 3.10)*

$$\begin{aligned} w_*((f\mathbf{X} \downarrow_n \mathbf{Z}) \xrightarrow{\pi} \mathbf{Z}) &= ((w_*f w_*\mathbf{X} \downarrow_n w_*\mathbf{Z}) \xrightarrow{\pi} w_*\mathbf{Z}) \in s^k \widehat{\mathbf{Cat}}, \quad \text{and} \\ w_*((f\mathbf{X} \downarrow_n g\mathbf{Y}) \xrightarrow{\pi} \mathbf{Y}) &= ((w_*f w_*\mathbf{X} \downarrow_n w_*g w_*\mathbf{Y}) \xrightarrow{\pi} w_*\mathbf{Y}) \in s^k \widehat{\mathbf{Cat}}. \end{aligned}$$

*Proof.* This follows readily from the observation that, for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$  (Df. 6.2)

$$w_{p_*}((\mathbf{Z} \downarrow_n \mathbf{Z}) \xrightarrow{\pi_0} \mathbf{Z}) = ((w_{p_*}\mathbf{Z} \downarrow_n w_{p_*}\mathbf{Z}) \xrightarrow{\pi_0} w_{p_*}\mathbf{Z}) \in \widehat{\mathbf{Cat}}. \quad \square$$

It remains the pull it all together as we promised in 6.1.

**6.7. Proposition.** *For every integer  $n \geq 1$ , every zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $\mathbf{Rel}^k \mathbf{Cat}$  or  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) can in a functorial manner be embedded in a commutative diagram of the form (Df. 6.4 and 6.5)*

$$\begin{array}{ccccc} \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} & \xrightarrow{k} & (f\mathbf{X} \downarrow_n g\mathbf{Y}) & \xrightarrow{\pi} & \mathbf{Y} \\ \downarrow & & \downarrow & & \downarrow g \\ \mathbf{X} & \xrightarrow{h} & (f\mathbf{X} \downarrow_n \mathbf{Z}) & \xrightarrow{\pi} & \mathbf{Z} \end{array}$$

in which  $h$  and  $k$  send each object  $X \in \mathbf{X}$  and  $(X, Y) \in \mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$  to the alternating zigzag of identity maps of  $fX$ , and

- (i) the squares are pullback squares, and
- (ii) the map  $h$  is a weak equivalence.

*Proof.* That the square on the right is a pullback square follows from Df. 6.4 and 6.5 and that the one on the left is so is a simple calculation.

That  $h$  is a weak equivalence follows readily from Pr. 6.3.  $\square$

## 7. PROPERTIES $B_n$ AND $C_n$

**7.1. Summary.** In final preparation for the formulation of our main results (in section 8) we recall from [DKS, §6] the notions of *properties*  $B_n$  and  $C_n$  in  $\widehat{\mathbf{Cat}}$  and then extend these notions to the categories  $\mathbf{Rel}^k \mathbf{Cat}$  and  $s^k \widehat{\mathbf{Cat}}$  for  $k \geq 1$ .

As these notions in  $\widehat{\mathbf{Cat}}$  are closely related to the *Grothendieck construction* we start with a discussion of the latter.

**7.2. Definition.** Given an object  $\mathbf{D} \in \widehat{\mathbf{Cat}}$  and a *not necessarily relative* functor  $F: \mathbf{D} \rightarrow \widehat{\mathbf{Cat}}$  (Df. 2.2(iii)), the **Grothendieck construction** on  $F$  is the object  $\mathbf{Gr} F \in \widehat{\mathbf{Cat}}$  which has

- (i) as *objects* the pairs  $(D, A)$  of objects  $D \in \mathbf{D}$  and  $A \in FD$ , and
- (ii) as *maps*  $(D_1, A_1) \rightarrow (D_2, A_2)$  the pairs  $(d, a)$  of maps

$$d: D_1 \longrightarrow D_2 \in \mathbf{D} \quad \text{and} \quad a: (FD_1)A_1 \longrightarrow A_2 \in FD_2$$

of which the *compositions* are given by the formula

$$(d', a')(d, a) = (d'd, a'((Fd)a)).$$

Such a Grothendieck construction  $\mathbf{Gr} F$  comes with a **projection functor**

$$\pi: \mathbf{Gr} F \longrightarrow \mathbf{D} \in \widehat{\mathbf{Cat}}$$

which sends an object  $(D, A)$  (resp. a map  $(d, a)$ ) to the object  $D$  (resp. the map  $d$ ) in  $\mathbf{D}$ .

The usefulness of Grothendieck constructions is due to the following property which was noted by Bob Thomason [T1, 1.2].

**7.3. Proposition.**

- (i) *The Grothendieck construction is a homotopy colimit functor on  $\widehat{\mathbf{Cat}}$  and hence*
- (ii) *it is homotopy invariant in the sense that every natural weak equivalence between two functors  $F_1, F_2: \mathbf{D} \rightarrow \widehat{\mathbf{Cat}}$  induces a weak equivalence  $\mathbf{Gr} F_1 \rightarrow \mathbf{Gr} F_2$ .*

**7.4. Example.** Given a map  $f: \mathbf{X} \rightarrow \mathbf{Z} \in \widehat{\mathbf{Cat}}$  and an integer  $n \geq 1$

- (i) denote, for every object  $Z \in \mathbf{Z}$  by (Df. 6.4)

$$(f\mathbf{X} \downarrow_n Z) \subset (f\mathbf{X} \downarrow_n \mathbf{Z}) \in \widehat{\mathbf{Cat}}$$

the category consisting of the objects and maps which end at  $Z$  or  $1_Z$ , and

- (ii) denote by

$$(f\mathbf{X} \downarrow_n -): \mathbf{Z} \longrightarrow \widehat{\mathbf{Cat}}$$

the *not necessarily relative* functor (Df. 2.2(iii)) which sends each object  $Z \in \mathbf{Z}$  to  $(f\mathbf{X} \downarrow_n Z)$  and each map  $z: Z \rightarrow Z' \in \mathbf{Z}$  to the functor  $(f\mathbf{X} \downarrow_n Z) \rightarrow (f\mathbf{X} \downarrow_n Z')$  obtained by “composition with  $z$ ”.

Then one readily verifies that

- (iii)  $(f\mathbf{X} \downarrow_n \mathbf{Z}) = \mathbf{Gr}(f\mathbf{X} \downarrow_n -)$ , and
- (iv)  $((f\mathbf{X} \downarrow_n \mathbf{Z}) \xrightarrow{\pi} \mathbf{Z}) = (\mathbf{Gr}(f\mathbf{X} \downarrow_n -) \xrightarrow{\pi} \mathbf{Z})$ .

**7.5. Example.** Given a map  $f: \mathbf{X} \rightarrow \mathbf{Z} \in \widehat{\mathbf{Cat}}$  and an integer  $n \geq 1$

- (i) denote, for every pair of objects  $X \in \mathbf{X}$  and  $Z \in \mathbf{Z}$ , by (Ex. 7.4)

$$(fX \downarrow_n Z) = (f\mathbf{X} \downarrow_n Z)$$

the category consisting of the objects and maps which start at  $fX$  or  $1_{fX}$ , and

- (ii) denote by

$$(f - \downarrow_n Z): \mathbf{X} \longrightarrow \widehat{\mathbf{Cat}} \quad \text{or} \quad (f - \downarrow_n Z): \mathbf{X}^{\text{op}} \longrightarrow \widehat{\mathbf{Cat}}$$

the functor which sends each object  $X \in \mathbf{X}$  to  $(fX \downarrow_n Z)$  and each map  $x: X \rightarrow X' \in \mathbf{X}$  to the induced functor

$$(fX \downarrow_n Z) \longrightarrow (fX' \downarrow_n Z) \quad \text{or} \quad (fX' \downarrow_n Z) \longrightarrow (fX \downarrow_n Z)$$

depending on whether  $n$  is even or odd.

Then one readily verifies that

(iii)

$$(fX \downarrow_n Z) = \begin{cases} \mathbf{Gr}((f - \downarrow_n Z): X \rightarrow \widehat{\mathbf{Cat}}) \\ \text{or} \\ \mathbf{Gr}((f - \downarrow_n Z): X^{\text{op}} \rightarrow \widehat{\mathbf{Cat}}) \end{cases}$$

### Properties $B_n$ and $C_n$ in $\widehat{\mathbf{Cat}}$

#### 7.6. Definition.

- (i) A map  $f: X \rightarrow Z \in \widehat{\mathbf{Cat}}$  is said to have **property**  $B_n$  ( $n \geq 1$ ) if the functor *Ex. 7.4*

$$(fX \downarrow_n -): Z \rightarrow \widehat{\mathbf{Cat}}$$

is a *relative functor* (Df. 2.2(iv)), and

- (ii) an object  $Z \in \widehat{\mathbf{Cat}}$  is said to have **property**  $C_n$  ( $n \geq 1$ ) if every map (Df. 2.9)

$$\mathbf{0}^w \rightarrow Z \in \widehat{\mathbf{Cat}}$$

has property  $B_n$ .

The usefulness of property  $C_n$  is due to the following result of [DKS, §6].

**7.7. Proposition.** *If, given a map  $f: X \rightarrow Z \in \widehat{\mathbf{Cat}}$ , the object  $Z$  has property  $C_n$  ( $n \geq 1$ ), then the map  $f: X \rightarrow Z$  has property  $B_n$ .*

*Proof.* In view of Ex. 7.4(ii) one has to show that every map  $z: Z \rightarrow Z' \in \mathbf{Z}$  induces a weak equivalence  $(fZ \downarrow_n Zj \rightarrow (fX \downarrow_n Z'))$ , or equivalently (Ex. 7.5(iii)) a weak equivalence

$$\mathbf{Gr}(f - \downarrow_n Z) \rightarrow \mathbf{Gr}(f - \downarrow_n Z')$$

But this follows readily from Ex. 7.5(iii) and Df. 7.5(ii) and the fact that, in view of property  $C_n$ , for every object  $X \in \mathbf{X}$  the map

$$(fX \downarrow_n Z) \rightarrow (fX \downarrow_n Z')$$

is a weak equivalence. □

### Properties $B_n$ and $C_n$ in $s^k \widehat{\mathbf{Cat}}$ and $\mathbf{Rel}^k \mathbf{Cat}$ ( $k \geq 1$ )

#### 7.8. Definition.

- (i) A map  $f: X \rightarrow Z \in s^k \mathbf{Cat}$  has **property**  $B_n$  ( $n \geq 1$ ) if
- for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$ , the map  $f_{p_*}: X_{p_*} \rightarrow Z_{p_*} \in \widehat{\mathbf{Cat}}$  has property  $B_n$  (Df. 7.6(i)),

and

- (ii) An object  $Z \in s^k \mathbf{Cat}$  has **property**  $C_n$  ( $n \geq 1$ ) if
- for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$  the object  $Z_{p_*} \in \widehat{\mathbf{Cat}}$  has property  $C_n$  (Df. 7.6(ii)).

### 7.9. Definition.

- (i) A map  $f: \mathbf{X} \rightarrow \mathbf{Z} \in \mathbf{Rel}^k \mathbf{Cat}$  has **property**  $B_n$  ( $n \geq 1$ ) if
  - the map  $w_* f: w_* \mathbf{X} \rightarrow w_* \mathbf{Z} \in s^k \widehat{\mathbf{Cat}}$  has property  $B_n$  (Df. 7.8(i)) or equivalently if
  - for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$  the map  $w_{p_*} f: w_{p_*} \mathbf{X} \rightarrow w_{p_*} \mathbf{Z} \in \widehat{\mathbf{Cat}}$  has property  $B_n$

and

- (ii) an object  $\mathbf{Z} \in \mathbf{Rel}^k \mathbf{Cat}$  has **property**  $C_n$  ( $n \geq 1$ ) if
  - the object  $w_* \mathbf{Z} \in s^k \widehat{\mathbf{Cat}}$  has property  $C_n$  (Df. 7.8(ii)) or equivalently if
  - for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$  the object  $w_{p_*} \mathbf{Z} \in \widehat{\mathbf{Cat}}$  has property  $C_n$ .

## Part III. The main results and their proofs

### 8. THE MAIN RESULTS

#### 8.1. Summary.

Our main results are

- (i) *Theorem 8.2* which states that the presence of properties  $B_n$  and  $C_n$  ensure that the potential homotopy pullbacks of section 6, i.e. the  $n$ -arrow *pullback objects*, are indeed *homotopy pullbacks*, and
- (ii) *Theorem 8.4* which states that the presence of a *strict 3-arrow calculus* implies *property*  $C_3$ .

These results then are applied to  $(\infty, 1)$ -categories and  $(\infty, k)$ -categories for  $k > 1$  to prove

- (iii) *Theorem 8.6* which combines Theorem 8.2 and 8.4 with results from [BK3] to show that *homotopy pullbacks in*  $(\infty, 1)$ -*categories* can be described as *3-arrow pullback objects* of zigzags between *partial model categories*, i.e. relative categories which have the *two out of six* property and admit a *3-arrow calculus*, and
- (iv) *Theorem 8.7* which is an application of Theorem 8.2 to  $(\infty, k)$ -categories for  $k \geq 1$ , which is much weaker than Theorem 8.6, because we have no model structure on  $\mathbf{Rel}^k \mathbf{Cat}$  for  $k \geq 1$ , nor an analog for partial model categories.

We also give proofs of Theorems 8.4, 8.6 and 8.7, but postpone the proof of Theorem 8.2 until section 10.

### The main results

**8.2. Theorem.** *Given an integer  $n \geq 1$ , let  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  be a zigzag in  $\mathbf{Rel}^k \mathbf{Cat}$  or  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) (Df. 3.6 and 3.9) with the property that*

- (i) *the map  $f: \mathbf{X} \rightarrow \mathbf{Z}$  has property  $B_n$  (Df. 7.6(i), 7.8(i) and 7.9(i))*

*which (Pr. 7.7) is in particular the case if*

- (ii) *the object  $\mathbf{Z}$  has property  $C_n$  (Df. 7.6(ii), 7.8(ii) and 7.9(ii)).*

*Then*

- (iii) *the  $n$ -arrow pullback object  $(f \mathbf{X} \downarrow_n g \mathbf{Y})$  (Df. 6.4 and 6.5) is a homotopy pullback (Df. 5.6 and 5.10) of this zigzag.*

The main tools for proving this are the *fibrillations* of Hopkins and Rezk which we will discuss in section 9 and we therefor postpone the proof of Theorem 8.2 until section 10.

**8.3. Corollary.** *A sufficient condition in order that the pullback  $\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y}$  of the above zigzag is also a homotopy pullback is that the obvious map*

$$\mathbf{X} \times_{\mathbf{Z}} \mathbf{Y} \xrightarrow{k} (f\mathbf{X} \downarrow_n g\mathbf{Y})$$

*of Pr. 6.7 is a weak equivalence.*

**8.4. Theorem.** *A sufficient condition in order that an object  $\mathbf{Z} \in \mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 0$ ) has property  $C_3$  (Df. 7.9(ii)) is that  $\mathbf{Z}$  admits a strict 3-arrow calculus (Df. 2.8 and 3.5).*

*Proof.*

- (i) The case  $k = 0$ . This follows from Rk. 2.7, Df. 2.8 and [DK, 3.3, 6.1, 6.2 and 8.2].
- (ii) The case  $k \geq 1$ . It follows readily from Df. 3.5 and 3.10 that, for every  $k$ -fold dimension  $p_* = (p_k, \dots, p_1)$ , the object  $w_{p_*} \mathbf{Z} \in \widehat{\mathbf{Cat}}$  admits a strict 3-arrow calculus and hence (i) has property  $C_3$ . The desired result then follows from Df. 7.9(ii).  $\square$

#### Applications to $(\infty, 1)$ -categories and $(\infty, k)$ -categories for $k > 1$

Before formulating the application mentioned in 8.1(iii) we recall some results from [BK3].

**8.5. Remark.** Recall from [BK3] that a **partial model category** is an object  $\mathbf{X} \in \mathbf{RelCat}$  which admits a 3-arrow calculus (Df. 2.5) and has the **two out of six property** that, for every three maps  $r, s$  and  $t \in \mathbf{X}$  for which the *two* compositions  $sr$  and  $ts$  exist and are weak equivalences, the other *four* maps  $r, s, t$  and  $tsr$  are also weak equivalences.

It then was shown in [BK3] that

- (i) *for every partial model category  $\mathbf{X} \in \mathbf{RelCat}$ , one (and hence every) Reedy fibrant approximation to  $\mathbf{sN} \mathbf{X} \in \mathbf{sS}$  (Df. 2.9) is a complete Segal space, i.e. a fibrant object in  $\mathbf{LsS}$  (Df. 4.5(i))*

which in view of [BK1, 6.1] implies that

- (ii)  *$\mathbf{X}$  is a quasi-fibrant object (Df. 5.7) of  $\mathbf{LRelCat}$  (Df. 4.5(iii)).*

Moreover

- (iii) *every complete Segal space is Reedy equivalent to the simplicial nerve of a partial model category.*

**8.6. Theorem.** If  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  is a zigzag in  $\mathbf{RelCat}$  in which

(i)  $\mathbf{X}$  and  $\mathbf{Y}$  are quasi-fibrant in  $\mathbf{LRelCat}$  and  $\mathbf{Z}$  is a partial model category, which in particular is the case if

- (ii)  $\mathbf{X}, \mathbf{Y}$  and  $\mathbf{Z}$  are all three partial model categories,

then

- (iii)  *$(f\mathbf{X} \downarrow_3 g\mathbf{Y})$  is a quasi-fibrant object of  $\mathbf{LRelCat}$ , which is a homotopy pullback of this zigzag not only in  $\mathbf{RelCat}$ , but also in  $\mathbf{LRelCat}$ .*

*Proof.* This follows readily from Rk.2.7, Pr. 5.8 and Th. 8.2 and 8.4.  $\square$

It remains to deal with the result that was mentioned in 8.1(iv).

**8.7. Theorem.** *Let  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  be a zigzag in  $\mathbf{Rel}^k \mathbf{Cat}$  ( $k \geq 1$ ) for which*

(i)  $s^k \mathbf{N} \mathbf{X}$ ,  $s^k \mathbf{N} \mathbf{Z}$  and  $s^k \mathbf{N} \mathbf{Y}$  *are quasi-fibrant objects of  $\mathbf{Ls}^k \mathbf{S}$  (Df. 4.5(i)) or equivalently (Pr. 4.4)*

(ii)  $w_* \mathbf{X}$ ,  $w_* \mathbf{Z}$  and  $w_* \mathbf{Y}$  *are quasi-fibrant objects of  $\mathbf{Ls}^k \widehat{\mathbf{Cat}}$  and assume that for some integer  $n \geq 1$*

(iii) *the map  $f: \mathbf{X} \rightarrow \mathbf{Z}$  has property  $B_n$  which (Pr. 7.7) in particular is the case if*

(iv) *the object  $\mathbf{Z}$  has property  $C_n$ , then*

(v) *( $f \mathbf{X} \downarrow_n g \mathbf{Y}$ ) is a homotopy pullback of this zigzag not only in  $\mathbf{Rel}^k \mathbf{Cat}$  but also in  $\mathbf{LRel}^k \mathbf{Cat}$  (Df. 4.5(iii)).*

*Proof.* In view of Df. 6.5 and 7.9, Pr. 6.6 and Th. 8.2  $w_*(f \mathbf{X} \downarrow_n g \mathbf{Y})$  is a homotopy pullback if the zigzag

$$w_* f: w_* \mathbf{X} \longrightarrow w_* \mathbf{Z} \longleftarrow w_* \mathbf{Y} : w_* g$$

in  $s^k \widehat{\mathbf{Cat}}$  as well as, in view of Pr. 5.8, in  $\mathbf{Ls}^k \widehat{\mathbf{Cat}}$ .

The desired result now follows readily from the *Global equivalence lemma* 5.13 and the fact that (Df. 4.2 and Pr. 4.4)  $w_*$  is a *homotopy equivalence*.  $\square$

## 9. HOPKINS-REZK FIBRILLATIONS

**9.1. Summary.** Our proof of Theorem 8.2 (in section 10) will consist of two parts. The first consists of a proof for the *model categories*  $s^k \widehat{\mathbf{Cat}}$  and  $\mathbf{RelCat}$ . In the second we lift these results for the model categories  $s^k \widehat{\mathbf{Cat}}$  ( $k > 1$ ) to the *relative categories*  $\mathbf{Rel}^k \mathbf{Cat}$  by means of the *Global equivalence lemma* 5.13.

Our aim in this section is to describe three lemmas which we will need for the first part. They involve, each in a different way, the *fibrillations* of Hopkins and Rezk as follows.

(i) The first lemma is a *Quillen fibrillation lemma* which will produce the fibrillations to get us started.

It is essentially a reformulation in terms of relative functors and fibrillations as well as a slight strengthening of the lemma that Quillen used to prove his Theorem B and states that

- given an object  $\mathbf{D} \in \widehat{\mathbf{Cat}}$ , a functor  $f: \mathbf{D} \rightarrow \widehat{\mathbf{Cat}}$  is a *relative functor* iff the projection functor from its Grothendieck construction to  $\mathbf{D}$  (Df. 7.2) is a *fibrillation* in  $\widehat{\mathbf{Cat}}$ .

(ii) The second lemma is a *Fibrillation lifting lemma* which enables us to obtain more fibrillation as it provides

- a sufficient condition on a relative functor in order that it *reflects fibrillations*.

(iii) The third lemma is a *Hopkins-Rezk fibrillation lemma* which shows how, in a right proper model category, some of the fibrillations obtained in (i) and (ii) can be used to construct homotopy pullbacks.



### Fibrillations

We start with recalling from [R2, §2] the notion of what Charles Rezk called *sharp maps* but which, because of their fibration-like properties (see 9.3 below), we prefer to call *fibrillations*.

**9.2. Definition.** Given a relative category  $\mathbf{R}$  with pullbacks (i.e. which is closed under pullbacks) a map  $p \in \mathbf{R}$  is called a **fibration** if every diagram in  $\mathbf{R}$  of the form

$$\begin{array}{ccccc} \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \end{array} \begin{array}{c} j \\ \\ p \\ \\ i \end{array}$$

in which

- the squares are pullback squares, and
- $i$  is a weak equivalence

has the property that

- $j$  is also a weak equivalence.

This definition readily implies that, just like the *fibrations* in a *right proper* model category, fibrillations have the following properties.

**9.3. Proposition.**

- (i) *Every pullback of a fibration is again a fibration, and*
- (ii) *every pullback of a weak equivalence along a fibration is again a weak equivalence.*

*Proof.* This is straightforward. □

### The fibration lifting lemma

**9.4. Lemma.** *Let  $f: \mathbf{R}_1 \rightarrow \mathbf{R}_2$  be a relative functor between relative categories with pullbacks which*

- *preserves pullbacks (e.g. is a right adjoint), and*
- *reflects weak equivalences.*

*Then*

- *it also reflects fibrillations.*

*Proof.* Given a pullback diagram in  $\mathbf{R}_1$  of the form

$$\begin{array}{ccccc} \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \xrightarrow{\quad} & \cdot & \xrightarrow{\quad} & \cdot \end{array} \begin{array}{c} j \\ \\ f \\ \\ i \end{array}$$

in which  $i$  is a weak equivalence and of which the image in  $\mathbf{R}_2$  is a similar diagram in which  $fj$  is a fibration. Then  $fj$  is a weak equivalence and hence so is  $j$ . □

**9.5. Example.** Examples of functors which satisfy the conditions of the fibration lifting lemma 9.4 are the abstract nerve functors of Df. 4.2 and Pr. 4.4.

### The Hopkins-Rezk fibration lemma

We recall from [R2, §2] the following

9.6. **Lemma.** *Let*

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{h} & B \xrightarrow{\pi} Z \end{array}$$

be a diagram in a right proper model category in which

- the square is a pullback square
- $h$  is a weak equivalence, and
- $\pi$  is a fibrillation.

Then  $A$  is a homotopy pullback of the zigzag

$$\pi: B \longrightarrow Z \longleftarrow Y : g$$

and hence (Df. 5.6) also of the zigzag

$$\pi h: X \longrightarrow Z \longleftarrow Y : g$$

### The Quillen fibrillation lemma

9.7. **Lemma.** *Given an object  $D \in \widehat{\mathbf{Cat}}$  (Df. 3.8(i)) and a functor  $F: D \rightarrow \widehat{\mathbf{Cat}}$  (Df. 2.2(iii)), the following three statements are equivalent.*

- (i)  $F$  is a relative functor (Df. 2.2(iv)),
- (ii) the map  $N\pi: N\mathbf{Gr} F \rightarrow N\mathbf{D} \in \mathbf{S}$  (Nt. 3.8(iii) and Df. 7.2) is a fibrillation, and
- (iii) the map  $\pi: \mathbf{Gr} F \rightarrow D \in \widehat{\mathbf{Cat}}$  is a fibrillation.

*Proof.* To prove (i) $\Rightarrow$ (ii) we note that

- Quillen's proof of this lemma [Q, §1] implies that, for every integer  $p \geq 0$  and map  $\Delta[p] = N\mathbf{p} \rightarrow N\mathbf{D}$ , the pullback

$$N\mathbf{p} \times_{N\mathbf{D}} N\mathbf{Gr} F$$

of the zigzag  $N\mathbf{p} \rightarrow N\mathbf{D} \leftarrow N\mathbf{Gr} F$  is a homotopy pullback

and that

- in view of [R2, 4.1(i and (ii))] this implies that the map  $N\pi \rightarrow N\mathbf{Gr} F \rightarrow N\mathbf{D} \in \mathbf{S}$  is a fibrillation.

That (ii) $\Rightarrow$ (iii) then follows from the *Fibrillation lifting lemma* 9.6 and Ex. 9.5.

Finally (iii) $\Rightarrow$ (i) follows by a simple calculation from the fact that, in view of the *Hopkins-Rezk fibrillation lemma* 9.6 the pullbacks of the form (Df. 2.8)

$$\mathbf{0}^w \times_D \mathbf{Gr} F \quad \text{and} \quad \mathbf{1}^w \times_D \mathbf{Gr} F$$

are both homotopy pullbacks. □

## 10. A PROOF OF THEOREM 8.2

10.1. **Preliminaries.** We start with recalling from Pr. 6.7 that

- (i) every zigzag  $f: \mathbf{X} \rightarrow \mathbf{Z} \leftarrow \mathbf{Y} : g$  in  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 0$ ) or  $\mathbf{RelCat}$  can, for every integer  $n \geq 1$ , in a functorial manner be embedded in a commutative diagram of the form (Df. 6.4 and 6.5)

$$\begin{array}{ccc} (f\mathbf{X} \downarrow_n g\mathbf{Y}) & \xrightarrow{\pi} & \mathbf{Y} \\ \downarrow & & \downarrow g \\ \mathbf{X} \xrightarrow{h} (f\mathbf{X} \downarrow_n \mathbf{Z}) & \xrightarrow{\pi} & \mathbf{Z} \end{array}$$

in which

- the square is a pullback square, and
- $h$  is a weak equivalence

and note that, in view of the *Hopkins-Rezk fibrillation lemma* 9.6

- (ii) if the map  $\pi: (f\mathbf{X} \downarrow_n \mathbf{Z}) \rightarrow \mathbf{Z}$  is a fibrillation, then the object  $(f\mathbf{X} \downarrow_n g\mathbf{Y})$  is a homotopy pullback of that zigzag.

10.2. **A proof for the category  $\widehat{\mathbf{Cat}}$ .**

- (i) It follows from Df. 7.2 and 7.6(i), Ex. 7.4(iii) and the *Quillen fibrillation lemma* 9.7 that the map  $\pi: (f\mathbf{X} \downarrow_n \mathbf{Z}) \rightarrow \mathbf{Z}$  is a fibrillation, and
- (ii) the desired result now follows from 10.1.

10.3. **A proof for the categories  $s^k \widehat{\mathbf{Cat}}$  ( $k \geq 1$ ).**

- (i) In view of 10.2(i) and Df. 7.8, for every  $k$ -fold dimension  $p_*$ , the map

$$\pi_{p_*}: (f_{p_*} \mathbf{X}_{p_*} \downarrow_n \mathbf{Z}_{p_*}) \longrightarrow \mathbf{Z}_{p_*} \in \widehat{\mathbf{Cat}}$$

is a fibrillation, and

- (ii) if  $\prod_{p_*} p_*$  denotes the product of these maps for all  $k$ -fold dimensions and  $\prod_{p_*} \widehat{\mathbf{Cat}}$  denotes the corresponding product of copies of  $\widehat{\mathbf{Cat}}$ , then clearly the same holds for the map

$$\prod_{p_*} (\pi_{p_*}: (f_{p_*} \mathbf{X}_{p_*} \downarrow_n \mathbf{Z}_{p_*}) \rightarrow \mathbf{Z}_{p_*}) \in \prod_{p_*} \widehat{\mathbf{Cat}}$$

- (iii) Moreover one readily verifies that, in view of Df. 6.5, the obvious map  $s^k \widehat{\mathbf{Cat}} \rightarrow \prod_{p_*} \widehat{\mathbf{Cat}}$  satisfies the conditions of the *Fibrillation lifting lemma* 9.4, which implies that the map  $\pi: (f\mathbf{X} \downarrow_n \mathbf{Z}) \rightarrow \mathbf{Z} \in s^k \widehat{\mathbf{Cat}}$  is also a fibrillation.
- (iv) The desired result now follows from 10.1.

10.4. **Two proofs for the category  $\mathbf{RelCat}$ .**

- (i) As the map  $f: \mathbf{X} \rightarrow \mathbf{Z} \in \mathbf{RelCat}$  has property  $B_n$  (by assumption), so does, in view of Df. 7.9, the map  $w_* f: w_* \mathbf{X} \rightarrow w_* \mathbf{Z} \in s^k \widehat{\mathbf{Cat}}$ .
- (ii) It follows from 10.3 that the map  $\pi: (w_* f w_* \mathbf{X} \downarrow w_* \mathbf{Z}) \rightarrow w_* \mathbf{Z} \in s^k \widehat{\mathbf{Cat}}$  is a fibrillation.
- (iii) Moreover as (Ex. 9.5)  $w_*$  satisfies the conditions of the *Fibrillation lifting lemma* 9.4 it follows from 6.6 that the map  $\pi: (f\mathbf{X} \downarrow_n \mathbf{Z}) \rightarrow \mathbf{Z} \in \mathbf{RelCat}$  is also a fibrillation and the desired result then follows from 10.1.

However if one does not want to use the model structure on **RelCat** one can also, instead of using (ii) and (iii) proceed as follows:

- (ii)' It follows from 10.3 that the object  $(w_*fw_*\mathbf{X} \downarrow w_*gw_*\mathbf{Y})$  is a homotopy pullback of the zigzag  $w_*f: w_*\mathbf{X} \rightarrow w_*\mathbf{Z} \leftarrow w_*\mathbf{Y} : w_*g$ , and
- (iii)' as (Pr. 4.4)  $w_*$  is a homotopy equivalence the desired result now follows from Pr. 6.6 and the *Global equivalence Lemma* 5.13.

10.5. **A proof for the categories  $\mathbf{Rel}^k\mathbf{Cat}$  ( $k > 1$ ).** This is essentially the same as the second proof for **RelCat** (10.4(i), (ii)' and (iii)').

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